Renormalization of the Φ^4 scalar theory under Robin boundary conditions and a possible new renormalization ambiguity

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We perform a detailed analysis of renormalization at one-loop order in the $\lambda\phi^4$ theory with Robin boundary condition (characterized by a constant c) on a single plate at $z = 0$. For arbitrary $c \geq 0$ the renormalized theory is finite after the inclusion of the usual mass and coupling constant counterterms, and two independent surface counterterms. A surface counterterm renormalizes the parameter c. The other one may involve either an additional wave-function renormalization for fields at the surface, or an extra quadratic surface counterterm. We show that both choices lead to consistent subtraction schemes at one-loop order, and that moreover it is possible to work out a consistent scheme with both counterterms included. In this case, however, they can not be independent quantities. We study a simple one-parameter family of solutions where they are assumed to be proportional to each other, with a constant ϑ. Moreover, we show that the renormalized Green functions at one-loop order does not depend on ϑ. This result is interpreted as indicating a possible new renormalization ambiguity related to the choice of ϑ.

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I. INTRODUCTION

Boundary conditions are of prime importance in the quantum domain. Observables, i.e. Hermitian operators in a Hilbert space, are defined only after its domain (as given by the set of boundary condition) is specified. This observation underlies the concept of self-adjoint extensions of an operator and has far reaching consequences in quantum physics [1, 2]. For a recent application of these ideas in a toy model for topology changing fluctuation in quantum gravity see [3].

Boundary conditions (BC) are ubiquitous in QFT for several reasons. A prototypical example is given by the Casimir effect, i.e. an observable force produced by zero-point vacuum fluctuations of a quantum field in the presence of boundaries or uncharged surfaces [4]. In this case the classical BC provides a simple mathematical model for the otherwise complex interactions between the quantum field and the classical body [5, 6]. More recently much attention has been paid to the question of BC in the context of the holographic principle and the brane-world scenario.

There is an ongoing effort to improve the calculation of the Casimir force arising from the fluctuations of electromagnetic field. This is necessary for a reliable comparison with recent high-precision measurements of the Casimir force between a flat plate and a spherical surface (lens) or a sphere [7], and also between two parallel flat plates [8]. Moreover, the Casimir effect has analogues in condensed matter physics, for instance in fluctuation induced forces [9] and critical phenomena in semi-confined systems [10], and may even play a key role in the design and operation of micro- and nano-scale electromechanical devices [11].

A field ϕ is said to obey Robin boundary condition at a surface Σ if its normal derivative at a point on Σ is proportional to its value there:

$$\frac{\partial}{\partial n}\,\phi(x) = c\,\phi(x), \qquad x \in \Sigma. \tag{1}$$

Neumann and Dirichlet boundary conditions arise as particular cases of Robin boundary condition: the first one corresponds to $c = 0$; the other is obtained in the limit $c \to \infty$ (assuming that $\partial_n\phi$ is bounded). In this paper we will restrict ourselves to the case $c \geq 0$.

The mixed case of Dirichlet-Robin BC were considered in [12] for a 2D massless scalar field as a phenomenological model for a penetrable surface, with c^{-1} playing the role of the finite penetration depth. Recently, the Casimir energy for a scalar field subject to Robin BC on one or two parallel planes (separated by a distance a) was computed in

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[13, 14]. There, it was shown that for the mixed case of Dirichlet-Robin BC the Casimir energy as a function of a develops a minimum, i.e., there is a configuration of stable equilibrium for the position of the planes (see also [15] for an interesting approach to the computation of the Casimir energy). Robin BC for scalar fields in the background of the Schwarzchild, de Sitter, and conformally flat Brane-World geometries have been investigated in [16], whereas the heat-kernel expansion for manifolds with Robin BC on the boundary was studied in [17]. Moreover, Robin BC are relevant in stabilization mechanisms of the compactification radius of large extra dimensions in five and six dimensional orbifolds [18].

In this paper we perform a detailed study of renormalization at one-loop order in the scalar $\lambda\phi^4$ in the presence of a flat surface at $z = 0$ for Robin BC. In contrast to the existing computations [10, 21] which pay attention to the particular cases $c = 0, \infty$ and use a mixed coordinate-momentum space regularization, we will keep $c \geq 0$ arbitrary and workout the regularization entirely in momentum space. This procedure avoids dealing with distributions and test functions otherwise unavoidable in the coordinate space regularization. Renormalization in the presence of hard boundaries [5] poses new difficulties related to the loss of full Lorentz invariance. Theories without Lorentz invariance raised a considerable theoretical and experimental interest in recent times, see for instance [19, 20]. This gives another motivation to our work, since again the BC leads to a simple model to study renormalization in Lorentz non-invariant theories.

The present study continues the one-loop renormalization which was started in [14], where the connected two-point function at first order in λ was considered. It was shown that in order to renormalize the two-point Green function one has to introduce two independent surface counterterms at the same order besides the usual (bulk, i.e without the surface at $z = 0$) mass counterterm at one-loop. A surface counterterm renormalizes the parameter c, and requires the introduction of a term of the form $\delta c \delta(z)\phi^2$ in the counterterm Lagrangian. The other demands the introduction of a new term

$$\delta b\, \delta(z)\, \phi\, (\partial_z - c)\, \phi\ . \tag{2}$$

Here we will consider the connected four-point Green function at second order in λ. In particular, it is shown that the renormalized four-point function is finite with the inclusion of the bulk coupling constant counterterm at order λ^2, together with the one-loop bulk mass counterterm and the two surface counterterms mentioned above. In other words, no extra ultraviolet (UV) singularities in the four-point function arises when the external points approach the surface. This is in agreement with the general framework developed in [10, 21].

Regarding the vertex (2) there is a kind of ordering ambiguity which appears when some or all external points approach the surface at $z = 0$ [10]. In [14] we followed the prescription of first let the external points approach the surface and later integrate over the vertex (2). This ordering will be called *AFIL* ("Approach First, Integrate Later"). On the other hand, it is possible to proceed in the other way round, namely first integrate over the vertex (2) and later approach the surface [10]. We call this prescription *IFAL* ("Integrate First, Approach Later"). The counterterm (2) makes no contribution at all with this ordering prescription since the factors involving δb cancel identically. Thus, in this case one is forced to implement another subtraction procedure. This is done by introducing an extra renormalization constant Z_s in order to allow for an additional renormalization of the surface field [10]. We show that both prescriptions lead to consistent subtraction schemes at one-loop order. In fact, it is possible to go one step further. Within the *AFIL* prescription, it is possible to include both counterterms ($\delta\, b$ and Z_s). However, they cannot be independent renormalization constants. We show that the simple parameterization $Z_s = \vartheta\, \delta\, b$ leads to a consistent renormalized theory at one-loop order. Moreover, we prove that the renormalized Green functions do not depend on the parameter ϑ. Thus, the ordering ambiguity in dealing with the vertex (2) seems to lead to a new type of renormalization ambiguity (involving the choice of ϑ), which was as far as we know unnoticed up to now.

The paper is organized as follows. In Section **II** we review some relevant material and fix the notation [14]. In Section **III** we revisit the computation of the renormalized two-point Green function at one-loop order, first using the prescription *AFIL*, and then with the prescription *IFAL*. In the same Section we investigate the choice $Z_s = \vartheta\, \delta\, b$. Section **IV** describes the renormalization of the connected four-point function for the parameterization $Z_s = \vartheta\, \delta\, b$. Finally, Section **V** is devoted to the conclusions and comments. Some useful formulas are collected in three Appendixes.

II. PERTURBATION THEORY FOR ROBIN BOUNDARY CONDITION

Consider a real scalar field in $D = d + 1$ dimensions living in the half-space $z \geq 0$ [24], with the Euclidean action

$$S[\phi] = \int d^d\mathbf{x} \int_0^\infty dz \left[\frac{1}{2} \left(\partial_\mu \phi\right)^2 + U(\phi) + c\, \phi^2\, \delta(z) \right], \tag{3}$$

The stationary action principle applied to the action (3) gives [25]

$$\delta S = \int d^d\mathbf{x} \left\{ (-\partial_z\phi + c\phi)\,\eta\Big|_{z=0} + \int_0^\infty dz\,[-\partial^2\phi + U'(\phi)]\,\eta \right\} + O(\eta^2). \tag{4}$$

From (4) one gets the (Euclidean) equation of motion

$$-\,\partial^2\phi + U'(\phi) = 0 \tag{5}$$

and the Robin boundary condition at $z = 0$,

$$\partial_z\phi - c\phi\Big|_{z=0} = 0. \tag{6}$$

Furthermore, $\phi(\mathbf{x},\infty) = 0$ and $\phi(\mathbf{x}\to\infty, z) = 0$.

The partial Fourier transform of $\phi(\mathbf{x}, z)$ is

$$\phi(\mathbf{x}, z) = \int \frac{d^d\mathbf{k}}{(2\pi)^d}\, e^{i\mathbf{k}\cdot\mathbf{x}}\,\varphi(\mathbf{k})\,e^{-kz}, \tag{7}$$

where $k = |\mathbf{k}|$. In this mixed $k - z$ representation, the Feynman propagator (unperturbed two-point Green function) reads [14] (the dependence on c will be omitted)

$$\mathcal{G}(\mathbf{k}; z, z') = \mathcal{G}^{\mathrm{D}}(\mathbf{k}; z, z') + \delta\mathcal{G}(\mathbf{k}; z, z') \tag{8}$$

where $(z_<(z_>) = \min(\max)\{z, z'\})$

$$\mathcal{G}^{\mathrm{D}}(\mathbf{k}; z, z') = \frac{1}{k}\,\sinh(kz_<)\,\exp(-kz_>), \tag{9}$$

is the Dirichlet propagator, $\mathcal{G}^{\mathrm{D}}(\mathbf{k}; 0, z') = \mathcal{G}^{\mathrm{D}}(\mathbf{k}; \infty, z') = 0$, and

$$\delta\mathcal{G}(\mathbf{k}; z, z') = \frac{e^{-k(z+z')}}{c+k}, \tag{10}$$

is a decreasing function of c. From (8)-(10) it is possible to obtain the alternative representation [10]

$$\mathcal{G}(\mathbf{k}; z, z') = \frac{1}{2k}\left[\, e^{-k\,|z-z'|} - \frac{c-k}{c+k}\, e^{-k\,(z+z')}\,\right], \tag{11}$$

where the first term inside the square brackets is the bulk (i.e. in the absence of the surface at $z = 0$) propagator of the scalar field. The Feynman propagator satisfies the Robin BC at $z = 0$,

$$\partial_z\,\mathcal{G}(\mathbf{k}; z, z')\Big|_{z=0} = c\,\mathcal{G}(\mathbf{k}; 0, z')\ , \tag{12}$$

and lacks full translational invariance due to the second term inside the square brackets in Eq. (11).

In the limit $z' \to 0$ one gets from (12)

$$\lim_{z'\to 0}\left[\lim_{z\to 0}\,\partial_z\,\mathcal{G}(\mathbf{k}; z, z')\right] = c\,\mathcal{G}(\mathbf{k}; 0, 0). \tag{13}$$

There is an ordering ambiguity in the double limit $z \to 0, z' \to 0$ [10], as can be seen by first letting $z' \to 0$, and then taking the derivative at $z = 0$,

$$\lim_{z\to 0}\left[\lim_{z'\to 0}\,\partial_z\,\mathcal{G}(\mathbf{k}; z, z')\right] = -k\,\mathcal{G}(\mathbf{k}; 0, 0). \tag{14}$$

The implementation of BC via local terms in the action is employed in studies of boundary critical phenomena [10]. In that context, it can be shown that Dirichlet and Neumann BC correspond to the so-called ordinary ($c \to \infty$) and special transitions ($c = 0$), respectively. The Robin BC is relevant in the study of the crossover between those universality classes, for which, however, the computations become much more involved. It is relevant also for the analysis of the ordinary transition; in this case, however, one may resort to an expansion in powers of c^{-1} [10].

It is convenient to distinguish between Green functions with and without points on the boundary. We then define the Green functions

$$G^{(N,L)}(x_1, ..., x_N; \mathbf{x}_{N+1}, ..., \mathbf{x}_L) := \langle\, \phi(x_1)...\phi(x_N)\phi_s(\mathbf{x}_{N+1})...\phi_s(\mathbf{x}_L)\,\rangle, \tag{15}$$

where the notation is $\phi_s(\mathbf{x}) = \phi(\mathbf{x}, 0)$. Initially, however, we will work with $G^{(M)}(x_1, ..., x_M)$ and later identify the Green functions $G^{(N,L)}$ by letting $M - N = L$ external points approach the boundary.

In the $\lambda\phi^4$ theory the only primitively UV divergent amplitudes are the two- and four-point Green functions[26]. This property is not spoiled by the Robin BC [10, 21]. In [14] it was shown that to renormalize the two-point Green function at first order in λ it is necessary to include three renormalization constants (δm^2, δc, and δb). We will revisit this calculation, paying special attention to the consequences of the ambiguity displayed in (13) and (14).

The regularized Euclidean Lagrangian density of the theory will be defined as

$$\mathcal{L}_0(\phi_0; m_0, \lambda_0, c_0, b_0; \Lambda) = \frac{1}{2}\,(\partial_\mu\phi_0)^2 + \frac{1}{2}\,m_0^2\phi_0^2 + \frac{\lambda_0}{4!}\,\phi_0^4 + c_0\,\delta(z)\,\phi_0^2 + b_0\,\delta(z)\,\phi_0\,\partial_z\phi_0, \tag{16}$$

with the field ϕ_0 living in the half-space $z \geq 0$, and where the dependence on the cut-off scale Λ is via the bare parameters $m_0,\ \lambda_0,\ c_0,\ b_0$ and bare field ϕ_0.

The finite, renormalized Green functions are given from

$$G_R^{(N,L)}(x_1, ..., \mathbf{x}_L; m, \lambda, c, b; \kappa) = \lim_{\Lambda\to\infty} Z^{-(N+L)/2}\, G^{(N,L)}(x_1, ..., \mathbf{x}_L; m_0, \lambda_0, c_0, b_0; \Lambda) \tag{17}$$

with $G^{(N,L)}$ the regularized Green functions computed from (16) and κ is a mass scale defining the subtraction point. In the multiplicative renormalization scheme

$$m_0^2 = Z^{-1}\,(m^2 + \delta m^2)\ , \tag{18}$$

$$\lambda_0 = Z^{-2}\,(\lambda + \delta\lambda), \tag{19}$$

$$c_0 = Z^{-1}\,(c + \delta c) - Z^{-1}\,c\,(b - \delta b) \tag{20}$$

$$b_0 = Z^{-1}\,(b + \delta b), \tag{21}$$

gives the relation between the renormalized and bare parameters. The renormalized field is $\phi = Z^{-1/2}\phi_0$, with $Z = 1 + \delta Z$. At $d = 3$ δm^2 contains both a logarithmic and a quadratic divergence. Thus it may be written as $\delta m^2 = \delta Z_m m^2 + \Delta m^2$. In an analogous way, one may write $\delta c = \delta Z_c c + \Delta c$. Both Δm^2 and Δc vanish in the dimensional regularization (DR) scheme.

In perturbation theory, the bare parameters and Z in Eqs. (18)-(21) are written as power series in λ so that, for $\delta f \in \{\delta Z,\ \delta m^2,\ \delta\lambda,\ \delta c,\ \delta b\}$,

$$\delta f = \sum_{k=1}^{\infty} \lambda^k\,\delta f_k\ , \tag{22}$$

where the coefficients $(\delta Z_k,\ \delta m_k^2,\ \delta\lambda_k,\ \delta c_k,\ \delta b_k)$ are functions of $m,\ c,\ b$ and Λ (or ϵ in DR). Its singular part is chosen so that the limit $\Lambda \to \infty$ ($\epsilon \to 0$ in DR) in (17) gives a well-defined function of the renormalized parameters and κ. The finite part is fixed by the renormalization conditions (RC).

The mass, coupling constant, and wave-function counterterms can be chosen as usual, namely using two conditions on the 1PI two-point function and one condition on the 1PI four-point function, both evaluated in the theory without the surface at $z = 0$ [10, 21]. Therefore, with this choice they take the same values of the bulk theory for a given subtraction scheme. In particular, $\delta Z_1 = \delta\lambda_1 = 0$. As for the surface renormalization constants, a natural choice for RC is

$$\mathcal{G}_R^{(0,2)}(\mathbf{k})\Big|_{k=\kappa} = (c + \kappa)^{-1}, \tag{23}$$

$$\frac{d}{dk}\mathcal{G}_R^{(0,2)}(\mathbf{k})\Big|_{k=\kappa} = -(c + \kappa)^{-2} \tag{24}$$

since they are satisfied at tree level, where $\mathcal{G}_R^{(0,2)}(\mathbf{k}) = \mathcal{G}(\mathbf{k}; 0, 0)$. Notice that we will set $m = 0$. For this reason we have to be careful when $k = 0$ is taken in $\mathcal{G}_R^{(0,2)}(\mathbf{k})$ and its derivatives, specially in the particular case $c = 0$. Moreover, one may employ a minimal subtraction scheme.

The defining relations (18)-(21) can be enforced directly on the Lagrangian density. We will study the particular case $m = b = 0$, so that

$$\mathcal{L}_0(\phi_0; m = 0, \lambda, c, b = 0) = \mathcal{L}^{(0)}(\phi_0; c) + \mathcal{L}_{\rm I}(\phi_0; c, \lambda) \ , \tag{25}$$

where we have split $\mathcal{L}_0$ into a free, unperturbed part,

$$\mathcal{L}^{(0)}(\phi_0; c) = \frac{1}{2}\,(\partial_\mu \phi_0)^2 + c\,\delta(z)\,\phi_0^2 \tag{26}$$

plus an interacting Lagrangian density, including counterterms. It is sufficient to consider only the connected parts of the two- and four-point Green functions. The interacting Lagrangian density which gives the connected parts of the two- and four-point Green functions at one-loop order reduces to

$$\mathcal{L}_{\rm I}(\phi_0; c, \lambda; \Lambda) = \frac{\lambda}{4!}\,\phi_0^4 + \frac{\lambda}{2}\,\delta m_1^2 \phi_0^2 + \frac{\lambda^2}{4!}\,\delta\lambda_2 \phi_0^4 + \lambda\delta c_1\,\delta(z)\,\phi_0^2 + \lambda\delta b_1\,\delta(z)\,\phi_0\,(\partial_z - c)\,\phi_0 + O(\lambda^2) \ . \tag{27}$$

The only counterterm of order λ^2 omitted in Eq. (27) is $(Z^{-1} - 1)c\,\delta(z)\,\phi_0^2$, the others being of high-order in λ. This is justified since it contributes to the two-point function only at two-loops, and to a disconnected part of the four-point function at one-loop.

Notice that we have chosen not to enforce the relation $\phi_0 = Z^{1/2}\phi$ on the Lagrangian level. Thus there remains an overall multiplicative renormalization, as displayed in Eq. (17).

III. TWO-POINT GREEN FUNCTION AT ONE-LOOP ORDER

The connected two-point Green function reads

$$\begin{aligned}\mathcal{G}^{(2)}(\mathbf{k}; z, z') = \mathcal{G}(\mathbf{k}; z, z') + \lambda \Bigg\{ & \delta\mathcal{G}_1(\mathbf{k}; z, z') - \delta m_1^2 \tilde{I}(\mathbf{k}; z, z') - \delta c_1 \mathcal{G}(\mathbf{k}; z, 0)\mathcal{G}(\mathbf{k}; 0, z') \\ & -\delta b_1 \int_0^\infty dz''\delta(z'')\Big[\mathcal{G}(\mathbf{k}; z, z'')\,(\partial_{z''} - c)\,\mathcal{G}(\mathbf{k}; z'', z') + (z \leftrightarrow z')\Big]\Bigg\} + O(\lambda^2),\end{aligned} \tag{28}$$

where

$$\tilde{I}(\mathbf{k}; z, z') := \int_0^\infty dz''\,\mathcal{G}(\mathbf{k}; z, z'')\,\mathcal{G}(\mathbf{k}; z'', z'), \tag{29}$$

and

$$\delta\mathcal{G}_1(\mathbf{k}; z, z') := -\frac{1}{2}\int \frac{d^d\mathbf{q}}{(2\pi)^d}\int_0^\infty dz''\,\mathcal{G}(\mathbf{k}; z, z'')\,\mathcal{G}(\mathbf{q}; z'', z'')\,\mathcal{G}(\mathbf{k}; z'', z') \ . \tag{30}$$

In Appendix **A** we show that for $z > 0,\ z' > 0$ one may expand $\delta\mathcal{G}_1(\mathbf{k}; z, z')$ as in Eq. (A.9),

$$\delta\mathcal{G}_1(\mathbf{k}; z, z') = -I_0\,\tilde{I}(\mathbf{k}; z, z') - J_0(0, c)\,\mathcal{G}(\mathbf{k}; z, 0)\,\mathcal{G}(\mathbf{k}; 0, z') + \Delta_1\mathcal{G}_1(\mathbf{k}; z, z') \ , \tag{31}$$

where $\Delta_1\mathcal{G}_1(\mathbf{k}; z, z')$ is regular for $z > 0,\ z' > 0$, and $d < 4$. Moreover,

$$J_n(k, c) := \frac{1}{8}\int \frac{d^d\mathbf{q}}{(2\pi)^d}\,\frac{1}{q^n(q + c)(q + k)} \ , \tag{32}$$

$$I_0 := \frac{1}{4}\int \frac{d^d\mathbf{q}}{(2\pi)^d}\,\frac{1}{q} \ . \tag{33}$$

From (28)-(31) we obtain

$$\begin{aligned}\mathcal{G}^{(2)}(\mathbf{k}; z, z') = \mathcal{G}(\mathbf{k}; z, z') + \lambda \Bigg\{ & \left(I_0 + \delta m_1^2 \right)\,\tilde{I}(\mathbf{k}; z, z') - [\,J_0(0, c) + \delta c_1\,]\,\mathcal{G}(\mathbf{k}; z, 0)\,\mathcal{G}(\mathbf{k}; 0, z') \\ & +\Delta_1\mathcal{G}_1(\mathbf{k}; z, z') - \delta b_1 \int_0^\infty dz''\delta(z'')\,[\mathcal{G}(\mathbf{k}; z, z'')(\partial_{z''} - c)\mathcal{G}(\mathbf{k}; z'', z') + (z \leftrightarrow z')]\Bigg\} + O(\lambda^2).\end{aligned} \tag{34}$$

For $z>0,\ z'>0$ the last term in (34) is zero due to the BC at $z''=0$, see (12). Moreover, adopting the RC ($p^\mu=(\mathbf{p},p_z)$ is a $d+1$ momentum)

$$\Gamma_R^{(2)}(p^2=0)=0, \tag{35}$$

on the 1PI two-point function of the bulk theory, one gets

$$\delta m_1^2=-I_0\ . \tag{36}$$

Now we make the identification $\mathcal{G}^{(2,0)}(\mathbf{k};z,z')=\mathcal{G}^{(2)}(\mathbf{k};z,z')$, and obtain from (17) the consistency condition

$$\lim_{\Lambda\to\infty}\ [\,J_0(0,c)+\delta c_1\,]=\text{finite function for } d=3\ . \tag{37}$$

This condition must be satisfied at $O(\lambda)$. Otherwise $\mathcal{G}_R^{(2,0)}$ will not be a regular function.

A. *AFIL* prescription

Now we will consider the limit $z\to 0$ of $G^{(2)}(\mathbf{k};z,z')$ with $z'\neq 0$. In this case, the ambiguity displayed in Eqs. (13) and (14) will show up in the evaluation of the last term on the RHS of (28),

$$\lim_{z\to 0}\int_0^\infty dz''\,\delta(z'')\,[\,\mathcal{G}(\mathbf{k};z,z'')\,(\partial_{z''}-c)\,\mathcal{G}(\mathbf{k};z'',z')+(z\leftrightarrow z')\,] \tag{38}$$

$$=-\frac{1}{2}\,\mathcal{G}(\mathbf{k};0,z'), \tag{39}$$

where to obtain (39) we followed the "ordering prescription"

AFIL: First attach the external point to the boundary and later integrate over the surface vertex.

This choice was adopted in [14]. For $z\to 0$ and $z'\neq 0$, the correct expansion of $\delta\mathcal{G}_1$ is given by Eq. (A.12) instead of Eq.(30). From (28), (A.12), (36), and (39) we obtain for $z'>0$

$$\begin{aligned}\lim_{z\to 0}\mathcal{G}^{(2)}(\mathbf{k};z,z') \;=\; &\mathcal{G}(\mathbf{k};0,z')-\lambda\Bigg\{\Bigg[\,J_0(\mathbf{k},c)+\frac{k-c}{2}\,J_1(k,c)-\frac{c\,(k+c)}{2}\,J_2(k,c)\\ &+\delta c_1-\frac{(c+k)}{2}\delta b_1\Bigg]\mathcal{G}(\mathbf{k};0,0)\,\mathcal{G}(\mathbf{k};0,z')-\Delta_2\mathcal{G}_1(\mathbf{k};z')\Bigg\}+O(\lambda^2)\ ,\end{aligned} \tag{40}$$

From now on we set $\mathcal{G}^{(1,1)}(\mathbf{k};z')=\lim_{z\to 0}\,\mathcal{G}^{(2)}(\mathbf{k};z,z')$. From (17) and (40) one gets

$$\lim_{\Lambda\to\infty}\left[\,J_0(k,c)+\frac{k-c}{2}\,J_1(k,c)+\delta c_1-\frac{k+c}{2}\delta b_1\right]=\text{finite fc. for } d=3\ . \tag{41}$$

If this condition is not satisfied the renormalized $\mathcal{G}_R^{(1,1)}(\mathbf{k};z)$ will not be a regular function at $O(\lambda)$.

Now consider the limits $z\to 0,\ z'\to 0$. This time, in place of (31) and (A.12) we will use Eq. (A.15). Moreover, in the *AFIL* prescription one gets

$$\lim_{z,z'\to 0}\int_0^\infty dz''\,\delta(z'')\,[\,\mathcal{G}(\mathbf{k};z,z'')\ (\partial_{z''}-c)\ \mathcal{G}(\mathbf{k};z'',z')+(z\leftrightarrow z')\,]\ =-\,\mathcal{G}(\mathbf{k};0,0)\ . \tag{42}$$

With (A.15) and (42), $\mathcal{G}_R^{(0,2)}(\mathbf{k})$ reads

$$\begin{aligned}\mathcal{G}_R^{(0,2)}(\mathbf{k}) \quad &=\lim_{\Lambda\to\infty}\mathcal{G}^{(2)}(\mathbf{k};0,0)=\mathcal{G}(\mathbf{k};0,0)\\ &-\lambda\lim_{\Lambda\to\infty}[\,J_0(k,c)-cJ_1(k,c)+\delta c_1-(c+k)\delta b_1\,]\ [\mathcal{G}(\mathbf{k};0,0)]^2+O(\lambda^2).\end{aligned} \tag{43}$$

Imposing the RC (23) and (24) upon $\mathcal{G}_R^{(0,2)}(\mathbf{k})$ gives us two equations whose solutions are

$$\delta b_1=J_0'(\kappa,c)-cJ_1'(0,c)\ , \tag{44}$$

$$\delta c_1=(c+\kappa)\,[\,J_0'(\kappa,c)-cJ_1'(\kappa,c)]-J_0(\kappa,c)+cJ_1(\kappa,c)\ , \tag{45}$$

where $J_n'(\kappa,c) = \partial J_n(k,c)/\partial k$ at $k=\kappa$.

Let us verify the consistency of this subtraction scheme. Using (44) and (45) in (37) and (41) it is easy to show that

$$\lim_{\Lambda\to\infty}\,[\,J_0(0,c)+\delta c_1\,] = -(c+\kappa)^2 \lim_{\Lambda\to\infty} J_1'(\kappa,c), \tag{46}$$

$$\lim_{\Lambda\to\infty}\left[\,J_0(k,c)+\frac{k-c}{2}\,J_1(k,c)+\delta c_1-\frac{k+c}{2}\delta b_1\right]$$
$$= -\frac{1}{2}\lim_{\Lambda\to\infty}\left[k(k+c)\,J_2(k,c)+(c+\kappa)(c+2\kappa)\,J_1'(\kappa,c)+\kappa^2(c+\kappa)\,J_2'(\kappa,c)\right]\ , \tag{47}$$

are regular functions for $d<4$. We conclude that the choice (36), (44) and (45) for δm_1^2, δb_1 and δc_1, respectively, lead to a well defined one-loop two-point Green function.

B. *IFAL* prescription

There is no a priori reason to adopt the ordering prescription *AFIL*. Instead it is quite conceivable to take the opposite route [10, 22], namely:

IFAL: First compute the integral over the surface vertex and later let the external point approach the surface.

In this case, using (13) we obtain in place of (39)

$$\lim_{z\to 0}\int_0^\infty dz''\,\delta(z'')\,[\,\mathcal{G}(\mathbf{k};z,z'')\,(\partial_{z''}-c)\,\mathcal{G}(\mathbf{k};z'',z')+(z\leftrightarrow z')\,]=0\ . \tag{48}$$

We conclude that the boundary counterterm (2) is ineffective in the *IFAL* prescription, and we may set $\delta b=0$ from the start. However, Eqs. (41) and (45) cannot be satisfied for $\delta b=0$.

In order to obtain the finite, renormalized Green functions one has to introduce an extra renormalization constant Z_s, to allow for an independent renormalization of the surface field [10],

$$\phi_0|_s = Z^{1/2}Z_s^{1/2}\phi(\mathbf{x},0) = Z_s^{1/2}\phi_0(\mathbf{x},0), \tag{49}$$

with $Z_s = 1+\delta Z_s$. Now, instead of (17) one defines (with $b=b_0=0$)

$$G_R^{(N,L)}(x_1,...,\mathbf{x}_L;m,\lambda,c;\kappa) = \lim_{\Lambda\to\infty} Z^{-(N+L)/2}Z_s^{-L/2}G^{(N,L)}(x_1,...,\mathbf{x}_L;m_0,\lambda_0,c_0;\Lambda) \tag{50}$$

with the multiplicative renormalization

$$c_0 = Z^{-1}Z_s^{-1}(c+\delta c) \tag{51}$$

replacing (20). Moreover, with (49) and (51) the vertex $\lambda\delta c_1\delta(z)\phi_0^2$ in Eq. (27) is replaced by

$$\lambda\,(\delta c_1 - c\delta Z_{s,1})\,\delta(z)\phi_0^2\ . \tag{52}$$

Notice that since $\delta Z_s \neq 0$ it is not only convenient but also necessary to distinguish between Green functions with and without $L=0$. If we repeat the previous analysis we will end up with two conditions and two equations to be satisfied. The two equations are solved to give

$$\delta Z_{s,1} = -J_0'(\kappa,c)+cJ_1'(0,c)\ , \tag{53}$$

$$\delta c_1 = \kappa\,[\,J_0'(\kappa,c)-cJ_1'(\kappa,c)]-J_0(\kappa,c)+cJ_1(\kappa,c)\ , \tag{54}$$

whereas the two consistency conditions are now

$$\lim_{\Lambda\to\infty}\,[\,J_0(0,c)+\delta c_1 - c\,\delta Z_{s,1}\,] = \text{finite function for } d=3\ , \tag{55}$$

$$\lim_{\Lambda\to\infty}\left[\,J_0(k,c)+\frac{k-c}{2}J_1(k,c)+\delta c_1+\frac{k-c}{2}\delta Z_{s,1}\right] = \text{finite fc. for } d=3\ , \tag{56}$$

Using Eqs.(53) and (54) one verifies that they are given by the RHS of Eqs.(46) and (47), respectively. We conclude that the renormalized two-point Green function at one-loop do not depend on the choice of the ordering prescription.

C. Mixed subtraction scheme: *AFIL*

Within the *AFIL* prescription it is possible to go one step further by introducing both counterterms, δb and δZ_s. In this case, with the definition (49), Eqs. (20) and (21) are replaced by ($b = 0$)

$$c_0 = Z^{-1} Z_s^{-1} (c + \delta c - c\delta b) \tag{57}$$
$$b_0 = Z^{-1} Z_s^{-1} \delta b. \tag{58}$$

Moreover,

$$G_R^{(N,L)}(x_1, ..., \mathbf{x}_L; m, \lambda, c, b; \kappa) = \lim_{\Lambda\to\infty} Z^{-(N+L)/2}\, Z_s^{-L/2}\, G^{(N,L)}(x_1, ..., \mathbf{x}_L; m_0, \lambda_0, c_0, b_0; \Lambda) \tag{59}$$

is the generalization of (17) and (50). Repeating the previous computations, we obtain that

$$\lim_{\Lambda\to\infty} \left[\, J_0(0, c) + \delta c_1 - c\delta Z_{s,1} \,\right] \ , \tag{60}$$
$$\lim_{\Lambda\to\infty} \left[\, J_0(k, c) + \frac{k-c}{2} J_1(k, c) + \delta c_1 + \frac{k-c}{2}\delta Z_{s,1} - \frac{k+c}{2}\delta b_1 \right] \ , \tag{61}$$

must be finite functions at $d = 3$. In addition, one gets the two equations

$$J_0(\kappa, c) - cJ_1(\kappa, c) + \delta c_1 - (c + \kappa)\delta b_1 + \kappa\delta Z_{s,1} = 0 \ , \tag{62}$$
$$\delta b_1 - \delta Z_{s,1} - J_0'(\kappa, c) + cJ_1'(\kappa, c) = 0 \ , \tag{63}$$

It is clear that δb_1 and $\delta Z_{s,1}$ can not be independent constants, since only two conditions, (23) and (24), are available. In other words, they are redundant renormalization constants. Let us assume that

$$\delta Z_s = \vartheta\delta b. \tag{64}$$

with $\vartheta \neq 1$ ($\vartheta = 1$ is ruled out by (63)). The solutions to (62), (63) and (64) can be cast in the form

$$\delta c_1 = \left(\kappa + \frac{c}{1-\vartheta}\right) \left[\, J_0'(\kappa, c; \Lambda) - cJ_1'(\kappa, c; \Lambda)\right] - J_0(\kappa, c; \Lambda) + cJ_1(\kappa, c; \Lambda) \ , \tag{65}$$
$$\delta b_1 = \frac{1}{1-\vartheta} \left[\, J_0'(\kappa, c; \Lambda) - cJ_1'(\kappa, c; \Lambda)\right] \ , \tag{66}$$
$$\delta Z_{s,1} = \frac{\vartheta}{1-\vartheta} \left[\, J_0'(\kappa, c; \Lambda) - cJ_1'(\kappa, c; \Lambda)\right] \ , \tag{67}$$

where we have explicitly indicated the dependence of J_0 and J_1 on the cut-off Λ, see Appendix **B**. The surface counterterms are functions of ϑ. Renormalized quantities however can not depend on the choice of ϑ. Indeed, using (65) and (66) one shows from (60) and (61) that they are given again by the RHS of (46) and (47), respectively. The choice $\delta Z_s = 0$ is parameterized by the limit $\vartheta \to 0$, and leads to the results (44) and (45). On the other hand, taking $\vartheta \to \infty$ reproduces (53) and (54), corresponding to the case $\delta b = 0$ in the *IFAL* prescription.

Using the results in Appendix **B** the counterterms may be written as ($d = 3$)

$$\delta c_1 = -\frac{1}{16\pi^2} \left[\Lambda - c\, \frac{k^2(1-2\vartheta) + c\,(c+2\kappa)}{(1-\vartheta)\,(c-\kappa)^2} \ln\left(\frac{\Lambda}{\kappa}\right) \ + \ 2c^2\, \frac{2\kappa\,(1-\vartheta) + c\vartheta}{(1-\vartheta)\,(c-\kappa)^2} \ln\left(\frac{\Lambda}{c}\right) \right] \ + \ \delta\bar{c}_1 \ , \tag{68}$$
$$\delta b_1 = -\frac{1}{16\pi^2} \frac{1}{1-\vartheta} \frac{1}{(c-\kappa)^2} \left[(\kappa^2 - 2c\,\kappa - c^2) \ln\left(\frac{\Lambda}{\kappa}\right) \ + \ 2c^2 \ln\left(\frac{\Lambda}{c}\right) \right] \ + \ \delta\bar{b}_1 \ , \tag{69}$$
$$\delta Z_{s,1} = -\frac{1}{16\pi^2} \frac{\vartheta}{1-\vartheta} \frac{1}{(c-\kappa)^2} \left[(\kappa^2 - 2c\,\kappa - c^2) \ln\left(\frac{\Lambda}{\kappa}\right) \ + \ 2c^2 \ln\left(\frac{\Lambda}{c}\right) \right] \ + \ \delta\bar{Z}_{s,1} \ , \tag{70}$$

where $\delta\bar{b}_1$, $\delta\bar{c}_1$, and $\delta\bar{Z}_{s,1}$ are finite contributions.

In [22] the surface counterterms were computed at second order in λ for the particular case $c = 0$ in the DR scheme. Moreover the *IFAL* prescription was implicitly assumed. This corresponds to the limit $\vartheta \to \infty$ of Eqs. (68)-(70). Notice that the counterterm δZ_s corresponds to δZ_1 in [22]. In the limit $\vartheta \to \infty$ and $c = 0$ we get from (70) (only the divergent part is displayed)

$$Z_s = 1 \ + \ \frac{\lambda}{16\pi^2} \ln\left(\frac{\Lambda}{\kappa}\right) \ + \ O(\lambda^2) \ , \tag{71}$$

This agrees with the result quoted in [22] after identifying $\ln(\Lambda/\kappa)$ with ϵ^{-1}. From (57) we get for c_0 (with $\delta c = c\,\delta Z_c + \Delta c$, $Z_c = 1 + \delta Z_c$, and $\delta b = 0$ for $\vartheta \to \infty$)

$$c_0 = Z^{-1}\, Z_s^{-1}\, Z_c\, c \;+\; Z^{-1}\, Z_s^{-1}\, \Delta c \;, \tag{72}$$

Our definition of Z_c is thus related to the analogous definition in [22] by

$$Z_c^{\mathrm{D}} = Z^{-1}\, Z_s^{-1}\, Z_c \;. \tag{73}$$

From (68) and (70) we obtain in the limit $\vartheta \to \infty$ and $c = 0$ (divergent part only)

$$Z_c^{\mathrm{D}} = 1 \;+\; \frac{\lambda}{16\pi^2} \ln\left(\frac{\Lambda}{\kappa}\right) \;+\; O(\lambda^2) \;, \tag{74}$$

which is the result quoted in [22] ($\ln(\Lambda/\kappa) \to \epsilon^{-1}$).

A remarkable feature of (68) is that the logarithmic divergent part of δc_1 depends on ϑ. In particular, for $c \to 0$ we have

$$Z_c = 1 \;-\; \frac{\lambda}{16\pi^2} \left(\frac{2\vartheta - 1}{1 - \vartheta}\right) \ln\left(\frac{\Lambda}{\kappa}\right) \;+\; O(\lambda^2) \;, \tag{75}$$

plus regular $O(\lambda)$ corrections. Therefore, in this case the choice $\vartheta = 1/2$ renders Z_c finite to one-loop order. Linear divergences are absent in the DR scheme, implying that $\Delta c = 0$. Hence in the DR scheme it seems possible to tune ϑ so as to eliminate *all* divergent terms from δc_1. Renormalization in this case is implemented by δb_1 and $\delta Z_{s,1}$.

Now the case $c \to \infty$, i.e. a Dirichlet BC on $z = 0$. It seems from (68)-(70) that the limit $c \to \infty$ is ill defined. However this is not true. Using the formulas given above it is possible to show that, in the case $c \to \infty$, the connected two-point Green function satisfies the Dirichlet BC at tree level and at one-loop order. Indeed, the Dirichlet BC is preserved at each order in perturbation theory [21]. Therefore from (34), (40), and (43), for instance, it is easy to see that *the surface divergences are harmless in the Dirichlet case*, and there is no reason whatsoever to introduce the surface counterterms. This can only happens for $c \to \infty$. For general c the renormalized two-point Green function *does not* satisfy the Robin BC at $z = 0$ because of the surface counterterms.

To conclude this Section, we remark that there is a kind of ambiguity in the possible choice (parameterized by ϑ) of δZ_s, δb, and δc. This ambiguity can not be fixed at first order in λ, unless a particular choice of the ordering is adopted in the evaluation of (38) (e.g. namely, the *IFAL* prescription). This is the procedure followed in studies of surface critical phenomena [10, 22]. Despite this, one may argue that the physical renormalized Green functions do not depend on the choice of ϑ. This has been verified explicitly at one-loop for the two point Green function. In the next Section we make a first step towards a high-order verification of this conjecture by studying the four-point Green function at the second order in λ.

IV. FOUR-POINT GREEN FUNCTION AT ONE-LOOP ORDER

The partial Fourier transformed four-point Green function may be written as

$$\mathcal{G}^{(4)}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4; z_1, z_2, z_3, z_4) = (2\pi)^d\, \delta^{(d)}\Big(\sum_{j=1}^{4} \mathbf{p}_j\Big)\, \tilde{\mathcal{G}}^{(4)}(\{\mathbf{p}_\ell\}, \{z_\ell\}), \tag{76}$$

where $\{\mathbf{p}_\ell\}$ is a shorthand for $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4$, and analogously for $\{z_\ell\}$. In the same notation, the connected four-point function at first order in 0 λ is

$$\tilde{\mathcal{G}}^{(4)[1]}(\{\mathbf{p}_\ell\}, \{z_\ell\}) = -\lambda \int_0^\infty dz\, \mathcal{G}(\mathbf{p}_1, z_1)\, \mathcal{G}(\mathbf{p}_2, z_2)\, \mathcal{G}(\mathbf{p}_3, z_3)\, \mathcal{G}(\mathbf{p}_4, z_4), \tag{77}$$

The second order corrections to the connected four-point function are shown in Figure (1). The one-loop amplitude given in Fig. (1a) reads

$$\begin{aligned} \tilde{\mathcal{G}}^{(\mathbf{s})}(\{\mathbf{p}_\ell\}, \{z_\ell\}) \;&=\; \frac{\lambda^2}{2} \int_0^\infty dz dz' \prod_{\ell=1}^{2} \mathcal{G}(\mathbf{p}_\ell; z_\ell, z) \prod_{j=3}^{4} \mathcal{G}(\mathbf{p}_j; z', z_j) \\ &\times \int \frac{d^d\mathbf{p}}{(2\pi)^d}\, \mathcal{G}(\mathbf{p}; z, z')\, \mathcal{G}(\mathbf{p} - \mathbf{s}; z', z) \;, \end{aligned} \tag{78}$$

where $\mathbf{s} = \mathbf{p}_1 + \mathbf{p}_2$. Diagrams (1b) and (1c) may be read off from (78) by exchanging $(\mathbf{p}_2, z_2) \leftrightarrow (\mathbf{p}_3, z_3)$, $\mathbf{s} \to \mathbf{t} = \mathbf{p}_1 + \mathbf{p}_3$, and $(\mathbf{p}_2, z_2) \leftrightarrow (\mathbf{p}_4, z_4)$, $\mathbf{s} \to \mathbf{u} = \mathbf{p}_1 + \mathbf{p}_4$, respectively.

A. Proof that $\tilde{\mathcal{G}}^{(\mathbf{s})}(\{\mathbf{p}_\ell\},\{z_\ell\})$ does not contain surface singularities

In the following we will show that the amplitudes given in Figs. (1a), (1b), and (1c) contain only bulk UV divergent terms. In other words, $\tilde{\mathcal{G}}^{(\beta)}(\{\mathbf{p}_\ell\},\{z_\ell\})$ ($\beta = \mathbf{s},\mathbf{t},\mathbf{u}$) does not develop UV divergences when any or all z_ℓ approach the surface at $z=0$. To prove that it is enough to consider the case $z_1 = z_2 = z_3 = z_4 = 0$. Eq. (78) may be written as a sum of four terms,

$$\tilde{\mathcal{G}}^{(\mathbf{s})}(\{\mathbf{p}_\ell\},\{0\}) = \frac{\lambda^2}{2}\sum_{k=1}^{4}\tilde{\mathcal{G}}_j^{(\mathbf{s})}(\{\mathbf{p}_\ell\},\{0\})\ , \tag{79}$$

where the integrand of $\tilde{\mathcal{G}}_1^{(\mathbf{s})}$, $\tilde{\mathcal{G}}_2^{(\mathbf{s})}$, $\tilde{\mathcal{G}}_3^{(\mathbf{s})}$, and $\tilde{\mathcal{G}}_4^{(\mathbf{s})}$ contains the factor $e^{-p|z-z'|-p'(z+z')}$, $e^{-p'|z-z'|-p(z+z')}$, $e^{-(p+p')(z+z')}$, and $e^{-(p+p')|z-z'|}$, respectively, and $p' = |\mathbf{p}-\mathbf{s}|$.

After the integrals over z and z' are performed $\tilde{\mathcal{G}}_1^{(\mathbf{s})}(\{\mathbf{p}_\ell\},\{0\})$ reads

$$\begin{aligned}\tilde{\mathcal{G}}_1^{(\mathbf{s})}(\{\mathbf{p}_\ell\},\{0\}) = &-\frac{1}{4}\prod_{\ell=1}^{4}\frac{1}{p_\ell+c}\int\frac{d^d\mathbf{p}}{(2\pi)^d}\frac{1}{p\,|\mathbf{p}-\mathbf{s}|}\left(\frac{c-|\mathbf{p}-\mathbf{s}|}{c+|\mathbf{p}-\mathbf{s}|}\right)\\ &\times\frac{2(p+|\mathbf{p}-\mathbf{s}|)+\sum_{k=1}^{4}p_k}{(p+|\mathbf{p}-\mathbf{s}|+p_1+p_2)(p+|\mathbf{p}-\mathbf{s}|+p_3+p_4)(2|\mathbf{p}-\mathbf{s}|+\sum_{k=1}^{4}p_k)}\end{aligned} \tag{80}$$

This term will give an UV finite contribution to $\tilde{\mathcal{G}}^{(\mathbf{s})}(\{\mathbf{p}_\ell\},\{0\})$ for $d<4$. The term $\tilde{\mathcal{G}}_2^{(\mathbf{s})}(\{\mathbf{p}_\ell\},\{0\})$ in Eq. (79) can be obtained from Eq. (80) by the exchange $p \leftrightarrow |\mathbf{p}-\mathbf{s}|$,

$$\begin{aligned}\tilde{\mathcal{G}}_2^{(\mathbf{s})}(\{\mathbf{p}_\ell\},\{0\}) = &-\frac{1}{4}\prod_{\ell=1}^{4}\frac{1}{p_\ell+c}\int\frac{d^d\mathbf{p}}{(2\pi)^d}\frac{1}{p\,|\mathbf{p}-\mathbf{s}|}\left(\frac{c-p}{c+p}\right)\\ &\times\frac{2(p+|\mathbf{p}-\mathbf{s}|)+\sum_{k=1}^{4}p_k}{(p+|\mathbf{p}-\mathbf{s}|+p_1+p_2)(p+|\mathbf{p}-\mathbf{s}|+p_3+p_4)(2p+\sum_{k=1}^{4}p_k)}\end{aligned} \tag{81}$$

In a similar way

$$\begin{aligned}\tilde{\mathcal{G}}_3^{(\mathbf{s})}(\{\mathbf{p}_\ell\},\{0\}) \;=\; &\frac{1}{4}\prod_{\ell=1}^{4}\frac{1}{p_\ell+c}\int\frac{d^d\mathbf{p}}{(2\pi)^d}\frac{1}{p\,|\mathbf{p}-\mathbf{s}|}\left(\frac{c-p}{c+p}\right)\left(\frac{c-|\mathbf{p}-\mathbf{s}|}{c+|\mathbf{p}-\mathbf{s}|}\right)\\ &\times\frac{1}{(p+|\mathbf{p}-\mathbf{s}|+p_1+p_2)(p+|\mathbf{p}-\mathbf{s}|+p_3+p_4)}\end{aligned} \tag{82}$$

This term also gives a regular contribution to $\tilde{\mathcal{G}}^{(\mathbf{s})}(\{\mathbf{p}_\ell\},\{0\})$ at $d<4$ for non-exceptional external momenta $\mathbf{s}\neq 0$. In the case $\mathbf{s}=0$ it reduces to

$$\tilde{\mathcal{G}}_3^{(\mathbf{s}=0)}(\{\mathbf{p}_\ell\},\{0\}) = \frac{1}{16}\prod_{\ell=1}^{4}\frac{1}{p_\ell+c}\int\frac{d^d\mathbf{p}}{(2\pi)^d}\frac{1}{p^2\,(p+p_1)^2}\left(\frac{c-p}{c+p}\right)^2 , \tag{83}$$

which is UV finite for $d<4$ as well as IR finite for $p_1\neq 0$. Notice that there is a possible IR problem for $p_1 = 0$. This however has nothing to do with the surface at $z=0$, since it also occurs in the bulk theory in the massless case considered here. Similar remarks applies to $\tilde{\mathcal{G}}_1^{(\mathbf{s}=0)}$ and $\tilde{\mathcal{G}}_2^{(\mathbf{s}=0)}$. Finally, the last contribution to $\tilde{\mathcal{G}}^{(\mathbf{s})}(\{\mathbf{p}_\ell\},\{0\})$ in Eq. (79) reads

$$\tilde{\mathcal{G}}_4^{(\mathbf{s})}(\{\mathbf{p}_\ell\},\{0\}) = \frac{1}{4\sum_{k=1}^{4}p_k}\prod_{\ell=1}^{4}\frac{1}{p_\ell+c}\int\frac{d^d\mathbf{p}}{(2\pi)^d}\frac{1}{p\,|\mathbf{p}-\mathbf{s}|}\left[\frac{1}{p+|\mathbf{p}-\mathbf{s}|+p_1+p_2}+\frac{1}{p+|\mathbf{p}-\mathbf{s}|+p_3+p_4}\right]. \tag{84}$$

This term contains an UV divergent part for $d\geq 3$. Noticing that

$$\int_0^\infty dz\prod_{\ell=1}^{4}\mathcal{G}(\mathbf{p}_\ell;0,z) = \frac{1}{4\sum_{k=1}^{4}p_k}\prod_{\ell=1}^{4}\frac{1}{p_\ell+c}\ , \tag{85}$$

and defining

$$\Lambda_{\mathbf{s}} := \frac{1}{8} \int \frac{d^d\mathbf{p}}{(2\pi)^d} \frac{1}{p\,|\mathbf{p}-\mathbf{s}|} \left[\frac{1}{p+|\mathbf{p}-\mathbf{s}|+p_1+p_2} + \frac{1}{p+|\mathbf{p}-\mathbf{s}|+p_3+p_4} \right] , \tag{86}$$

one may write Eq. (84) as

$$\tilde{\mathcal{G}}_4^{(\mathbf{s})}(\{\mathbf{p}_\ell\},\{0\}) = \Lambda_{\mathbf{s}} \int_0^\infty dz \prod_{\ell=1}^4 \mathcal{G}(\mathbf{p}_\ell;0,z) \; . \tag{87}$$

Altogether, Eq. (79) may be cast in the form

$$\tilde{\mathcal{G}}^{(\mathbf{s})}(\{\mathbf{p}_\ell\},\{0\}) = \lambda^2 \left(\Lambda_{\mathbf{s}} + \delta\Lambda_{\mathbf{s}} \right) \int_0^\infty dz \prod_{\ell=1}^4 \mathcal{G}(\mathbf{p}_\ell;0,z) \; , \tag{88}$$

where $\delta\Lambda_{\mathbf{s}}$ contains the regular ($d<4$) contributions given in Eqs. (80)-(82).

The other "channels" ($\mathbf{t}$ and $\mathbf{u}$) give similar results. Therefore

$$\sum_{\beta=\mathbf{s},\mathbf{t},\mathbf{u}} \tilde{\mathcal{G}}^{(\beta)}(\{\mathbf{p}_\ell\},\{0\}) = \lambda^2 \left\{ \sum_{\beta=\mathbf{s},\mathbf{t},\mathbf{u}} \Lambda_\beta + \sum_{\beta=\mathbf{s},\mathbf{t},\mathbf{u}} \delta\Lambda_\beta \right\} \int_0^\infty dz \prod_{\ell=1}^4 \mathcal{G}(\mathbf{p}_\ell;0,z) \; , \tag{89}$$

The UV divergences are localized in the contributions given by Λ_β ($\beta = \mathbf{s},\ \mathbf{t},\ \mathbf{u}$). They are of logarithmic nature at $d=3$. It is clear that they are cancelled by the contribution coming from Fig. (1d), which is of the same form,

$$\mathcal{G}_{\rm d}(\{\mathbf{p}_\ell\},\{0\}) = -\lambda^2\,\delta\lambda_2 \int_0^\infty dz \prod_{\ell=1}^4 \mathcal{G}(\mathbf{p}_\ell;0,z) \; . \tag{90}$$

Now, in Appendix **C** is shown that the analogue of Eq. (89) for the case without the surface at $z=0$ is (see Eq. (C.7))

$$\sum_{\beta=\mathbf{s},\mathbf{t},\mathbf{u}} \tilde{\mathcal{G}}_B^{(\beta)}(\{\mathbf{p}_\ell\},\{0\}) = \lambda^2 \left\{ \sum_{\beta=\mathbf{s},\mathbf{t},\mathbf{u}} \Lambda_\beta + \sum_{\beta=\mathbf{s},\mathbf{t},\mathbf{u}} \delta\Lambda_{\beta,B} \right\} \int_0^\infty dz \prod_{\ell=1}^4 \mathcal{G}_B(\mathbf{p}_\ell;0,z) \; , \tag{91}$$

with *the same* Λ_β given above, and a finite ($d<4$) contribution $\delta\Lambda_{\beta,B}$.

Therefore, since the divergent part of Eqs. (91) (in the absence of the surface) and (89) (surface at $z=0$) are exactly equal, we conclude that (i) There are no surface divergences in $\tilde{\mathcal{G}}^{(\beta)}(\{\mathbf{p}_\ell\},\{0\})$ ($\beta=\mathbf{s},\mathbf{t},\mathbf{u}$); and (ii) The counterterm $\delta\lambda_2$ in Eq. (90) may be chosen equal to the one-loop coupling constant counterterm $\delta\lambda_2$ of the bulk theory. Moreover, the above procedure can be repeated for any number of external points z_ℓ outside of the surface at $z=0$. The conclusions are the same, the only difference being in the regular terms (for $d<4$) $\delta\Lambda_\beta$ in Eq. (89).

B. The other one-loop amplitudes

In the previous section we studied graphs (a)-(d) in Fig.(1). To complete the discussion, we will discuss the contributions associated with one-loop corrections to the external legs of $\mathcal{G}^{(4)[1]}$ in Eq. (77). These are given by the diagrams (e)-(t) in Fig.(1). In the following we will briefly discuss the case $z_1=z_2=z_3=z_4=0$. The case with some or all external points outside of the surface can be analyzed in the same way. Moreover, we will work with the mixed subtraction scheme of section **III.C** since it allows the inclusion of both δb and δZ_s. For $z_1=z_2=z_3=z_4=0$, graphs (e)-(h) read

$$\tilde{\mathcal{G}}_{\rm e}(\{\mathbf{p}_\ell\},\{0\}) = -\lambda \int_0^\infty dz\, \delta\mathcal{G}_1(\mathbf{p}_1;0,z) \prod_{\ell=2}^4 \mathcal{G}(\mathbf{p}_\ell;z,0) \; , \tag{92}$$

$$\tilde{\mathcal{G}}_{\rm f}(\{\mathbf{p}_\ell\},\{0\}) = \lambda^2\,\delta m_1^2 \int_0^\infty dz'\, \mathcal{G}(\mathbf{p}_1;0,z') \int_0^\infty dz\, \mathcal{G}(\mathbf{p}_1;z',z) \prod_{\ell=2}^4 \mathcal{G}(\mathbf{p}_\ell;z,0) \; , \tag{93}$$

$$\tilde{\mathcal{G}}_{\rm g}(\{\mathbf{p}_\ell\},\{0\}) = 2\lambda^2\Delta c_1 \int_0^\infty dz'\,\delta(z')\mathcal{G}(\mathbf{p}_1;0,z') \int_0^\infty dz\,\mathcal{G}(\mathbf{p}_1;z',z) \prod_{\ell=2}^4 \mathcal{G}(\mathbf{p}_\ell;z,0)\ , \tag{94}$$

$$\tilde{\mathcal{G}}_{\rm h}(\{\mathbf{p}_\ell\},\{0\}) = 2\lambda^2\delta b_1 \int_0^\infty dz\,\prod_{\ell=2}^4 \mathcal{G}(\mathbf{p}_\ell;z,0) \int_0^\infty dz'\,\delta(z')\,\mathcal{G}(\mathbf{p}_1;0,z')\ (\partial_{z'}-c)\ \mathcal{G}(\mathbf{p}_1;z',z)\ . \tag{95}$$

where $\Delta c_1 = \delta c_1 - c\delta Z_{s,1}$. The contributions from graphs (i)-(l), (m)-(p), and (q)-(t) may be obtained from (92)-(95) by the replacements $\mathbf{p}_1 \leftrightarrow \mathbf{p}_2$, $\mathbf{p}_1 \leftrightarrow \mathbf{p}_3$, and $\mathbf{p}_1 \leftrightarrow \mathbf{p}_4$, respectively.

It is known that for Robin BC the 1PI functions are not multiplicatively renormalized. This means that 1PR graphs must be included in a modified skeleton expansion [10, 21]. However, the connected Green functions are multiplicatively renormalized. We have shown that the amplitudes given by graphs (a)-(c) contain only bulk UV divergences. Thus surface divergences in the four-point function may arise only in connection with graph (e). Clearly no new surface divergences are introduced beside the ones discussed in Section **III**. These divergences are eliminated by the one-loop counterterm insertions in graphs (g) and (h). This has been verified using the results from Section **III**.

In this way one obtains the renormalized four-point function at one-loop,

$$\begin{aligned}
\tilde{\mathcal{G}}_R^{(0,4)}(\{\mathbf{p}_\ell\}) \ &=\ \lim_{\Lambda\to\infty} \left(1+\lambda\delta Z_{s,1}+O(\lambda^2)\right)^{-2} \Bigg[\tilde{\mathcal{G}}^{(0,4)[1]}(\{\mathbf{p}_\ell\}) + \sum_{\beta=\mathbf{s},\mathbf{t},\mathbf{u}} \tilde{\mathcal{G}}^{(\beta)}(\{\mathbf{p}_\ell\},\{0\}) \\
&\qquad + \sum_{\alpha={\rm e},\ldots,{\rm u}} \tilde{\mathcal{G}}_\alpha(\{\mathbf{p}_\ell\},\{0\}) + O(\lambda^3)\Bigg] \\
&=\ \tilde{\mathcal{G}}^{(0,4)[1]}(\{\mathbf{p}_\ell\}) + \lim_{\Lambda\to\infty}\lambda^2\left[\sum_{\beta=\mathbf{s},\mathbf{t},\mathbf{u}}\Lambda_\beta - \delta\lambda_2\right]\int_0^\infty dz\,\prod_{\ell=1}^4\mathcal{G}(\mathbf{p}_\ell;z,0) \\
&\quad + \lim_{\Lambda\to\infty}\lambda^2\sum_{k=1}^4\Bigg[\delta c_1 - c\delta Z_{s,1} - c\delta b_1 + J_0(p_k,c) + \frac{p_k-c}{2}\,(\,J_1(p_k,c)-\delta b_1\,) \\
&\quad + \frac{1}{2}(c+p_k)\delta Z_{s,1}\Bigg]\,\mathcal{G}(\mathbf{p}_k;0,0)\,\int_0^\infty dz\,\prod_{\ell=1}^4\mathcal{G}(\mathbf{p}_\ell;z,0) + \Delta\tilde{\mathcal{G}}^{(0,4)}(\{\mathbf{p}_\ell\}) + O(\lambda^3)\ ,
\end{aligned} \tag{96}$$

where $\Delta\tilde{\mathcal{G}}^{(0,4)}(\{\mathbf{p}_\ell\})$ includes $O(\lambda^2)$ regular contributions for $d<4$. As we have shown in section **IV.A**, the logarithmic (at $d=3$) UV divergence in Λ_β can be eliminated by a convenient choice of RC in the bulk theory. Therefore, the second term on the RHS of (96) is finite in the limit $\Lambda\to\infty$. As for the third term on the RHS, the following condition has to be met ($k=1,...,4$)

$$\lim_{\Lambda\to\infty}\left[\delta c_1 - c\delta Z_{s,1} - c\delta b_1 + J_0(p_k,c) + \frac{p_k-c}{2}\,(\,J_1(p_k,c)-\delta b_1\,) + \frac{1}{2}(c+p_k)\delta Z_{s,1}\right] = \text{finite}\ , \tag{97}$$

for $d<4$. This is the equation given in (61), and it is satisfied as shown in section **III.C**, with $\delta Z_{s,1}$, δb_1 and δc_1 given in Eqs. (65)-(67).

We conclude that the renormalized connected four-point function does not depend on the choice of ϑ, since the RHS of (97) has no dependence on ϑ as shown in section **III.C**. Therefore, $\mathcal{G}^{(0,4)}(\{\mathbf{p}_\ell\})$ is finite up to $O(\lambda^3)$ with the choice of *four* independent counterterms: a set of two independent surface counterterms,

$$\{\delta c_1(\vartheta), \delta b_1(\vartheta), \delta Z_{s,1}(\vartheta)\}, \quad \text{for } \vartheta\neq 1 \tag{98}$$

in addition to the two bulk counterterms δm_1^2 and $\delta\lambda_2$.

Finally we remark that the inclusion of another flat surface at $z=a$ does not change the overall picture presented here. Suppose that at $z=a$ the field satisfies a Robin BC parameterized by another constant $\tilde{c}$. Surface divergences localized on $z=a$ arise, but they are of the same type of the ones appearing in the case of a single flat surface at $z=0$. Therefore the new set of surface counterterms associated to the surface at $z=a$ is given by Eqs. (65)-(67) upon the replacement $c\to\tilde{c}$. In other words, surface counterterms of a new type are not required.

V. CONCLUSIONS

Boundary conditions are a key ingredient in quantum theory. It can be shown for instance that not all BC are consistent with the conservation of probability in quantum mechanics [1, 2]. Possible BC include periodic, Dirichlet,

Neumann, Robin, and other BC without a classical interpretation. Boundary conditions play a major role in the Casimir effect. The presence of surfaces or boundaries in the vacuum leads to a manifestation of the universal zero-point fluctuations intrinsic to any quantum system in the form of the Casimir force.

Renormalization theory in the presence of surfaces is hampered by difficulties associated with the loss of full Lorentz invariance. However, a number of results as well as a general framework may be developed for some BC [10, 21]. In this paper we performed a detailed renormalization analysis of the $\lambda\phi^4$ theory at one-loop order in the case of Robin BC at a single surface at $z = 0$. In contrast to previous investigations [10, 21] we keep the parameter $c \geq 0$ arbitrary and workout the regularization entirely in momentum space. Moreover, attention has been paid to the consequences of the ambiguity in the choice of the ordering prescription when dealing with insertions of the counterterm vertex (2).

As far as we know, the prescription of first integrate over the surface vertex and then let the external point approach the surface (*IFAL* prescription) is always tacitly assumed in the existing literature [10, 22]. In this case, the surface vertex (2) makes no contribution at all, and one is forced to introduce another counterterm related to an extra renormalization of the surface fields, see Eq.(49). However, we argued that the other prescription, e.g. first attach the external point to the surface and then integrate over the vertex (*AFIL* prescription), also leads to a well defined renormalized theory at one-loop order. In fact, working with the *AFIL* prescription it is possible to include both counterterms δb and δZ_s. They cannot of course be independent quantities. We have investigated a one-parameter family of solutions where $\delta Z_s = \vartheta \delta b$, and found that it leads to a consistent subtraction scheme at one-loop order. Moreover, we have shown that the renormalized two- and four-point connected Green functions do not depend on the choice of ϑ at one-loop order. We interpret this result as indicating a possible new renormalization ambiguity, related to the choice of ϑ.

Indeed, since the renormalized Green functions do not depend on ϑ one immediately obtains the equation

$$\begin{aligned} \frac{d}{d\vartheta}\,\mathcal{G}_R^{(N,L)}\left(\{\mathbf{p}_\ell\},\{z_\ell\}\right) \;&=\; 0 \\ &= \lim_{\Lambda\to\infty}\,\frac{d}{d\vartheta}\,Z^{-\frac{N+L}{2}}\,Z_s^{-\frac{L}{2}}\,\mathcal{G}^{(N,L)}\left(\{\mathbf{p}_\ell\},\{z_\ell\};m_0,\lambda_0,c_0,b_0;\Lambda\right)\ , \end{aligned} \tag{99}$$

where on the RHS the bare Green functions are computed from (16). For instance, assuming that the bare Green functions depend on ϑ only through the surface counterterms, one obtains for the case $N = 0$, $L = 2$ a self-consistency condition,

$$\left(\sigma_c\,\frac{\partial}{\partial c_0} + \sigma_b\,\frac{\partial}{\partial b_0}\right)\,\mathcal{G}^{(0,2)}(\mathbf{p}) = \sigma_s\,\mathcal{G}^{(0,2)}(\mathbf{p})\ , \tag{100}$$

where

$$\begin{aligned} \sigma_c &\equiv \frac{\partial c_0}{\partial\vartheta}\ ; \\ \sigma_b &\equiv \frac{\partial b_0}{\partial\vartheta} = -\frac{1}{c}\,\sigma_c + O\left(\lambda^2\right)\ ; \\ \sigma_s &\equiv \frac{\partial \ln Z_s}{\partial\vartheta} = -\frac{1}{c}\,\sigma_c + O\left(\lambda^2\right)\ . \end{aligned} \tag{101}$$

It is easy to show that condition (100) is satisfied at one-loop order. On the other hand, it is possible to turn (99) into a renormalization-group-like equation, for instance assuming that the bare Green functions bear an explicit dependence on the parameter ϑ.

It is conceivable however that a high-order calculation may rule out some values of ϑ as inconsistent. This remains to be verified, and for this reason we intend to carry our computation of the surface counterterms to the next order of perturbation theory. We are also investigating the application of the framework developed here to some problems in surface critical phenomena, in particular to the crossover between the ordinary and the special universality classes. It seems worth to explore Eq. (99) in this context.

Note: After the completion of this work we became aware of the paper [23] where a different Lagrangian quantization scheme for boundary quantum field theory is developed. There, the authors start from the Neumann BC and then define the more general interacting theory as a perturbation around this free theory. They also show how the Robin BC can be obtained in this way. It would be interesting to find the relation between the approach pursued here and the one in [23].

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APPENDIX A

Let us define

$$I(\mathbf{k},\mathbf{q};z,z') := \int_0^\infty dw\, e^{-2qw}\, \mathcal{G}_0(\mathbf{k};z,w)\, \mathcal{G}_0(\mathbf{k};w,z'). \tag{A.1}$$

Notice that $I(\mathbf{k},\mathbf{q};z,z') = I(\mathbf{k},\mathbf{q};z',z)$. Using the propagator (11) and doing the integral over w one obtains,

$$I(\mathbf{k},\mathbf{q};z,z') = I_1(\mathbf{k},\mathbf{q};z,z') + I_2(\mathbf{k},\mathbf{q};z,z')\ , \tag{A.2}$$

where

$$\begin{aligned} I_1(\mathbf{k},\mathbf{q};z,z') = \frac{1}{2(2k)^2}\Bigg\{ &\left(\frac{c-k}{c+k}\right)\frac{k}{q(q+k)}\, e^{-k(z+z')-2qz} \\ &+\theta(z-z')\, e^{-k(z-z')}\left[\frac{1}{q}\left(e^{-2qz'}-e^{-2qz}\right)+\left(\frac{e^{-2qz}}{q+k}-\frac{e^{-2qz'}}{q-k}\right)\right]\Bigg\} + (z\leftrightarrow z')\ , \end{aligned} \tag{A.3}$$

and

$$I_2(\mathbf{k},\mathbf{q};z,z') = \frac{1}{(2k)^2}\,\frac{e^{-k(z+z')}}{(k+c)^2}\left[\frac{(c+k)^2}{2(q-k)}+\frac{(c-k)^2}{2(q+k)}-\frac{(c+k)(c-k)}{q}\right] \tag{A.4}$$

Both I_1 in Eq. (A.3) and I_2 in Eq. (A.4) are symmetric functions of $z,\ z'$.

From the definition of $\delta\mathcal{G}_1(\mathbf{k};z,z')$ in Eq. (30) one gets

$$\begin{aligned} \delta\mathcal{G}_1(\mathbf{k};z,z') &= -I_0\,\tilde{I}(\mathbf{k};z,z') - \frac{1}{2}\int\frac{d^d\mathbf{q}}{(2\pi)^d}\left(\frac{1}{c+q}-\frac{1}{2q}\right) I(\mathbf{k},\mathbf{q};z,z') \\ &= -I_0\,\tilde{I}(\mathbf{k};z,z') - \frac{1}{2}\int\frac{d^d\mathbf{q}}{(2\pi)^d}\left(\frac{1}{c+q}-\frac{1}{2q}\right) I_1(\mathbf{k},\mathbf{q};z,z') \\ &\quad -\frac{1}{2}\int\frac{d^d\mathbf{q}}{(2\pi)^d}\left(\frac{1}{c+q}-\frac{1}{2q}\right) I_2(\mathbf{k},\mathbf{q};z,z')\ , \end{aligned} \tag{A.5}$$

where $\tilde{I}(\mathbf{k};z,z')$ and I_0 are defined in Eqs. (29) and (33), respectively. Now, it is clear that due to the exponential terms e^{-qz} and $e^{-qz'}$ in $I_1(\mathbf{k},\mathbf{q};z,z')$, the integral over $\mathbf{q}$ in the second term of the RHS of (A.5) is convergent, unless $z=0$ and/or $z'=0$. Assume for the moment that $z>0$ and $z'>0$. Thus, the only divergent contributions to $\delta\mathcal{G}_1(\mathbf{k};z,z')$ comes from the term containing $I_2(\mathbf{k},\mathbf{q};z,z')$, apart from the first term on the RHS of (A.5) which gives the (usual) bulk correction to the one-loop self-energy.

In order to get the leading divergent terms in the third term of (A.5) let us expand $I_2(\mathbf{k},\mathbf{q};z,z')$. For large q one obtains

$$I_2(\mathbf{k},\mathbf{q};z,z') \overset{q\to\infty}{=} \frac{e^{-k(z+z')}}{(c+k)^2}\left[\frac{1}{2q}+\frac{c}{2q^2}+O\left(\frac{1}{q^3}\right)\right]. \tag{A.6}$$

Using (A.6) inside the integrand of the third term on the RHS of (A.5) one gets its behavior for large $\mathbf{q}$,

$$\begin{aligned} -\frac{1}{2}\int\frac{d^d\mathbf{q}}{(2\pi)^d}\left(\frac{1}{c+q}-\frac{1}{2q}\right) I_2(\mathbf{k},\mathbf{q};z,z') &\sim -\frac{1}{2}\frac{e^{-k(z+z')}}{(c+k)^2}\int\frac{d^d\mathbf{q}}{(2\pi)^d}\left[\frac{1}{4q^2}-\frac{c}{4q^3}+O\left(\frac{1}{q^4}\right)\right] \\ &\sim -\frac{1}{2}\frac{e^{-k(z+z')}}{(c+k)^2}\int\frac{d^d\mathbf{q}}{(2\pi)^d}\left[\frac{1}{4q(q+c)}+O\left(\frac{1}{q^4}\right)\right] \\ &= -J_0(0,c)\,\mathcal{G}(\mathbf{k};z,0)\,\mathcal{G}(\mathbf{k};0,z') + (\text{regular terms in the UV})\ , \end{aligned} \tag{A.7}$$

where $J_0(k;c)$ was defined in Eq. (32). This means that

$$\overline{\Delta}\mathcal{G}_1(\mathbf{k};z,z') := -\frac{1}{2}\int \frac{d^d\mathbf{q}}{(2\pi)^d}\left(\frac{1}{c+q}-\frac{1}{2q}\right) I_2(\mathbf{k},\mathbf{q};z,z') + J_0(0,c)\,\mathcal{G}(\mathbf{k};z,0)\,\mathcal{G}(\mathbf{k};0,z')\ , \tag{A.8}$$

is a regular function for $d<4$ and *any value* of $z,\ z'$. Finally, we write Eq. (A.5) as

$$\delta\mathcal{G}_1(\mathbf{k};z,z') = -I_0\,\tilde{I}(\mathbf{k};z,z') - J_0(0,c)\,\mathcal{G}(\mathbf{k};z,0)\,\mathcal{G}(\mathbf{k};0,z') + \Delta_1\mathcal{G}_1(\mathbf{k};z,z')\ , \tag{A.9}$$

where

$$\Delta_1\mathcal{G}_1(\mathbf{k};z,z') = -\frac{1}{2}\int \frac{d^d\mathbf{q}}{(2\pi)^d}\left(\frac{1}{c+q}-\frac{1}{2q}\right) I_1(\mathbf{k},\mathbf{q};z,z') + \overline{\Delta}\mathcal{G}_1(\mathbf{k};z,z')\ , \tag{A.10}$$

is a regular function for $z>0$, $z'>0$, and $d<4$.

Consider now the case $z=0$ and $z'>0$. From Eqs. (A.2)-(A.4) we obtain

$$I(\mathbf{k},\mathbf{q};0,z') = \frac{1}{4}\left[\frac{c+k+2q}{q(q+k)} - \frac{c+k}{q(q+k)}\,e^{-2qz'}\right]\mathcal{G}(\mathbf{k};0,0)\,\mathcal{G}(\mathbf{k};0,z')\ . \tag{A.11}$$

Using Eq. (A.11) in Eq.(30) leads after some manipulations to the result

$$\delta\mathcal{G}_1(\mathbf{k};0,z') = -\left[\frac{1}{2k}I_0 + J_0(k,c) + \frac{k-c}{2}\,J_1(k,c) - \frac{c\,(k+c)}{2}\,J_2(k,c)\right]\mathcal{G}(\mathbf{k};0,0)\,\mathcal{G}(\mathbf{k};0,z') + \Delta_2\mathcal{G}_1(\mathbf{k};z')\ , \tag{A.12}$$

where

$$\Delta_2\mathcal{G}_1(\mathbf{k};z') = \frac{c+k}{8}\int \frac{d^d\mathbf{q}}{(2\pi)^d}\,e^{-2qz'}\,\frac{1}{q(q+k)}\left(\frac{1}{c+q}-\frac{1}{2q}\right)\mathcal{G}(\mathbf{k};0,0)\,\mathcal{G}(\mathbf{k};0,z') \tag{A.13}$$

gives a regular contribution for $z'\neq 0$.

Finally, consider the case $z=0$ and $z'=0$. Again, from Eqs. (A.2)-(A.4) one gets

$$I(\mathbf{k},\mathbf{q};0,0) = \frac{1}{2(q+k)}\,\left[\mathcal{G}(\mathbf{k};0,0)\right]^2\ . \tag{A.14}$$

Substituting Eq. (A.14) in Eq. (30) leads to

$$\delta\mathcal{G}_1(\mathbf{k};0,0) = -\left[\frac{1}{2k}I_0 + J_0(k,c) - cJ_1(k,c)\right]\,\left[\,\mathcal{G}(\mathbf{k};0,0)\,\right]^2. \tag{A.15}$$

APPENDIX B

Here we discuss the computation of

$$J_n(k,c) = \frac{1}{8}\int \frac{d^d\mathbf{q}}{(2\pi)^d}\,\frac{1}{q^n(q+c)(q+k)}\ , \tag{B.1}$$

using a regularization cutoff $\Lambda>0$ at $d=3$. By definition, the regularized version of Eq. (B.1) is

$$J_n(k,c;\Lambda) = \frac{1}{8}\int_{-\Lambda}^{\Lambda} \frac{d^3\mathbf{q}}{(2\pi)^3}\,\frac{1}{q^n(q+c)(q+k)}\ , \tag{B.2}$$

Employing the identity (valid for $A\neq B$)

$$\frac{1}{(q+A)(q+B)} = \frac{1}{B-A}\left[\frac{1}{q+A}-\frac{1}{q+B}\right], \tag{B.3}$$

one can recursively compute $J_n(k,c;\Lambda)$,

$$J_n(k,c;\Lambda) = \frac{1}{c-k}\,\left[\,J_{n-1}(k,0;\Lambda) - J_{n-1}(0,c;\Lambda)\,\right]\ , \tag{B.4}$$

for $n \geq 1$ from $J_0(k,c;\Lambda)$. This may be written from (B.2) and (B.3) as

$$J_0(k,c;\Lambda) = \frac{1}{8(c-k)} \int_{-\Lambda}^{\Lambda} \frac{d^3\mathbf{q}}{(2\pi)^3} \left[\frac{1}{q+k} - \frac{1}{q+c} \right]$$
$$= \frac{1}{16\pi^2} \left[\Lambda + \frac{k^2}{c-k} \ln\left(\frac{\Lambda}{k}\right) - \frac{c^2}{c-k} \ln\left(\frac{\Lambda}{c}\right) + O(\Lambda^{-1}) \right] . \quad \text{(B.5)}$$

From (B.4) and (B.5) one gets

$$J_1(k,c;\Lambda) = \frac{1}{16\pi^2} \frac{1}{c-k} \left[-k \ln\left(\frac{\Lambda}{k}\right) + c \ln\left(\frac{\Lambda}{c}\right) + O(\Lambda^{-1}) \right] . \quad \text{(B.6)}$$

Although the identity (B.3) is only valid for $A \neq B$, that is $k \neq c$, the limit $k \to c$ of (B.5) and (B.6) is well-defined,

$$J_0(c,c;\Lambda) = \frac{1}{16\pi^2} \left[\Lambda - 2c \ln\left(\frac{\Lambda}{c}\right) + O(\Lambda^{-1}) \right] ,$$
$$J_1(c,c;\Lambda) = \frac{1}{16\pi^2} \left[\ln\left(\frac{\Lambda}{c}\right) + O(\Lambda^{-1}) \right] . \quad \text{(B.7)}$$

Since $J_2(k,c)$ is regular in the UV one may take the limit $\Lambda \to \infty$. The integral may be computed from (B.1) or from (B.4), with the same result

$$J_2(k,c) = \frac{1}{16\pi^2} \frac{1}{c-k} \ln\left(\frac{c}{k}\right) . \quad \text{(B.8)}$$

valid for $k \neq c$. In the case $k = c$ the integral gives

$$J_2(c,c) = \frac{1}{16\pi^2} \frac{1}{c} . \quad \text{(B.9)}$$

Finally, the functions $J_n'(k,c) = \partial J_n(k,c)/\partial k$ can be computed along the same lines, or using the explicit formulas for $J_0(k,c;\Lambda)$, etc. .

APPENDIX C

In this Appendix we will show that the joint contribution of graphs (a)-(c) in Figure 1 to the one-loop connected four-point function of the $\lambda\phi^4$ theory without the surface at $z=0$ (bulk theory) can be written as in Eq. (91). Graph (a) may be obtained from (78) upon replacing the Robin propagator in (11) by the bulk propagator

$$\mathcal{G}_B(\mathbf{p}; z, z') = \frac{1}{2p} e^{-p|z-z'|} , \quad \text{(C.1)}$$

in the partial Fourier transform representation (7). It is enough to consider the case $z_1 = z_2 = z_3 = z_4 = 0$, since the bulk theory enjoy full translational symmetry. Then after some manipulations it is possible to break down $\tilde{\mathcal{G}}^{(\mathbf{s})}_B(\{\mathbf{p}_\ell\},\{0\})$ in two parts,

$$\tilde{\mathcal{G}}^{(\mathbf{s})}_B(\{\mathbf{p}_\ell\},\{0\}) = \frac{\lambda^2}{2} \left[\tilde{\mathcal{G}}^{(\mathbf{s})}_{1,B}(\{\mathbf{p}_\ell\},\{0\}) + \tilde{\mathcal{G}}^{(\mathbf{s})}_{2,B}(\{\mathbf{p}_\ell\},\{0\}) \right] . \quad \text{(C.2)}$$

The term $\tilde{\mathcal{G}}^{(\mathbf{s})}_{1,B}(\{\mathbf{p}_\ell\},\{0\})$ is regular for $d < 4$,

$$\tilde{\mathcal{G}}^{(\mathbf{s})}_{1,B}(\{\mathbf{p}_\ell\},\{0\}) = -\frac{1}{2} \prod_{\ell=1}^{4} \frac{1}{2p_\ell} \int \frac{d^d\mathbf{p}}{(2\pi)^d} \frac{1}{p\,|\mathbf{p}-\mathbf{s}|} \int_0^\infty dz\,dz'\, e^{-(p_1+p_2)z-(p_3+p_4)z'}\, e^{-(p+|\mathbf{p}-\mathbf{s}|)(z+z')}$$
$$= -\frac{1}{2} \prod_{\ell=1}^{4} \frac{1}{2p_\ell} \int \frac{d^d\mathbf{p}}{(2\pi)^d} \frac{1}{p\,|\mathbf{p}-\mathbf{s}|\,(p+|\mathbf{p}-\mathbf{s}|+p_1+p_2)\,(p+|\mathbf{p}-\mathbf{s}|+p_3+p_4)} . \quad \text{(C.3)}$$

The other contribution contains a logarithmic divergence for $d=3$,

$$\tilde{\mathcal{G}}^{(\mathbf{s})}_{2,B}(\{\mathbf{p}_\ell\},\{0\}) = \frac{1}{2}\prod_{\ell=1}^{4}\frac{1}{2p_\ell}\int\frac{d^d\mathbf{p}}{(2\pi)^d}\,\frac{1}{p\,|\mathbf{p}-\mathbf{s}|}\int_0^\infty dz\,dz'\,e^{-(p_1+p_2)z-(p_3+p_4)z'}\,e^{-(p+|\mathbf{p}-\mathbf{s}|)|z-z'|}$$

$$= \Lambda_{\mathbf{s}}\int_{-\infty}^{\infty}dz\,\prod_{\ell=1}^{4}\mathcal{G}_B(\mathbf{p}_\ell;0,z)\;, \tag{C.4}$$

where $\Lambda_{\mathbf{s}}$ was defined in Eq.(86), and we have used the identity

$$\int_{-\infty}^{\infty}dz\,\prod_{\ell=1}^{4}\mathcal{G}_B(\mathbf{p}_\ell;0,z) = \frac{2}{\sum_{k=1}^{4}p_k}\prod_{\ell=1}^{4}\frac{1}{2p_\ell}\;. \tag{C.5}$$

Altogether, Eq. (C.2) may be written as

$$\tilde{\mathcal{G}}^{(\mathbf{s})}_{B}(\{\mathbf{p}_\ell\},\{0\}) = \lambda^2\,(\,\Lambda_{\mathbf{s}}+\delta\Lambda_{\mathbf{s},B}\,)\int_{-\infty}^{\infty}dz\,\prod_{\ell=1}^{4}\mathcal{G}_B(\mathbf{p}_\ell;0,z)\;, \tag{C.6}$$

where $\delta\Lambda_{\mathbf{s},B}$ contains the regular ($d<4$) contributions given in Eq. (C.3). The other crossed graphs (b) and (c) give similar contributions and one may write

$$\sum_{\beta=\mathbf{s},\mathbf{t},\mathbf{u}}\tilde{\mathcal{G}}^{(\beta)}_{B}(\{\mathbf{p}_\ell\},\{0\}) = \lambda^2\left\{\sum_{\beta=\mathbf{s},\mathbf{t},\mathbf{u}}\Lambda_\beta+\sum_{\beta=\mathbf{s},\mathbf{t},\mathbf{u}}\delta\Lambda_{\beta,B}\right\}\int_{-\infty}^{\infty}dz\,\prod_{\ell=1}^{4}\mathcal{G}_B(\mathbf{p}_\ell;0,z)\;. \tag{C.7}$$

The more general case with $z_\ell\neq 0$ preserves the structure displayed in Eq. (C.6), with the same $\Lambda_{\mathbf{s}}$ on the RHS.

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[24] Conventions: $\hbar = c = 1$, $x = (\mathbf{x}, z)$, $\mathbf{x} = (x^0, \ldots, x^{d-1})$.
[25] Notice that $\int_0^\infty \delta(z)\, f(z)\, dz = \frac{1}{2}\, f(0)$.
[26] Composite operators will not be considered in this paper.

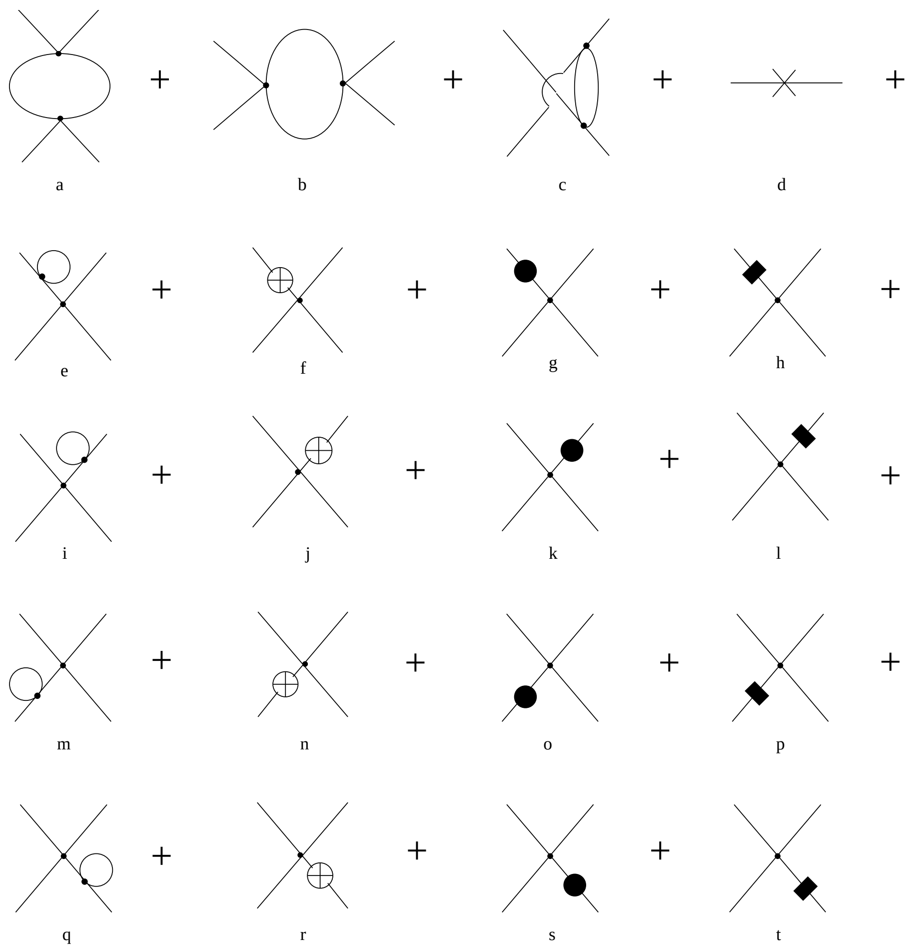

FIG. 1: Radiative corrections to the connected four-point Green function at one-loop. Notation: cross in (d) denotes $\delta\lambda_1$; crossed circle in (f) represent δm_1; black circle in (g) represent the vertex (52); and black box in (h) stand for the vertex (2).